

# ON PLURI-CANONICAL SYSTEMS OF ARITHMETIC SURFACES

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**ABSTRACT.** Let  $S$  be a Dedekind scheme with perfect residue fields at closed points, let  $f : X \rightarrow S$  be a minimal regular arithmetic surface of fibre genus at least 2 and let  $f' : X' \rightarrow S$  be the canonical model of  $f$ . It is well known that  $\omega_{X'/S}$  is relatively ample. In this paper we prove that  $\omega_{X'/S}^{\otimes n}$  is relatively very ample for all  $n \geq 3$ .

## 1. INTRODUCTION

Let  $S$  be a Dedekind scheme, let  $f : X \rightarrow S$  be a minimal regular arithmetic surface of (fibre) genus at least 2 (see [16], §9.3), and let  $\omega_{X/S}$  be the relative dualizing invertible sheaf on  $X$ . It is well known that  $\omega_{X/S}^{\otimes n}$  is globally generated when  $S$  is affine and  $n$  is big enough (actually  $n \geq 2$  suffices, see [14], Theorem 7 and Remark 1.3 below). But in general  $\omega_{X/S}$  is not relatively ample even when  $X \rightarrow S$  is semi-stable, because of the possible presence of vertical  $(-2)$ -curves (*i.e.*, vertical curves  $C$  in  $X$  such that  $\omega_{X/S} \cdot C = 0$ ). Let  $f' : X' \rightarrow S$  be the canonical model of  $X$  obtained by contracting all vertical  $(-2)$ -curves ([16], §9.4.3). Then the relative dualizing sheaf  $\omega_{X'/S}$  is a relatively ample invertible sheaf on  $X'$  (op. cit., §9.4.20).

**Question 1.1** For which  $n \in \mathbb{N}$  is  $\omega_{X'/S}^{\otimes n}$  relatively very ample?

When  $f$  is smooth, then  $X = X'$  and it is well known that  $\omega_{X'/S}^{\otimes n}$  is relatively very ample for all  $n \geq 3$ . More generally, the same result holds if  $f$  is semi-stable (hence  $X' \rightarrow S$  is stable), see [9], Corollary of Theorem 1.2. In this paper we show that the same bound holds in the general case under the mild assumption that the residue fields of  $S$  at closed points are perfect (we do not know whether this condition can be removed). Notice that this condition is satisfied if  $S$  is the spectrum of the ring of integers of a number field  $K$ , or if  $S$  is a regular, integral quasi-projective curve over a perfect field.

**Theorem 1.2 (Main Theorem).** *Let  $S$  be a Dedekind scheme with perfect residue fields at closed points, let  $f : X \rightarrow S$  be a minimal regular arithmetic surface of fibre genus at least 2, and let  $f' : X' \rightarrow S$  be the canonical model of  $f$ . Then  $\omega_{X'/S}^{\otimes n}$  is relatively very ample for all  $n \geq 3$ .*

**Remark 1.3** Under the above hypothesis and when  $S$  is affine, we also give a new proof of a theorem of J. Lee ([14], Theorem 7):

$\omega_{X/S}^{\otimes n}$  (equivalently,  $\omega_{X'/S}^{\otimes n}$ ) is globally generated for all  $n \geq 2$ .

The proof in [14] is based on a detailed case-by-case analysis. Our method is more synthetic. Moreover, we do not assume that the generic fibre of  $f$  is geometrically

irreducible, see Proposition 3.4. However, as a counterpart, we need the assumption on the perfectness of the residue fields.

The proof of 1.2 relies on the following criterion due to F. Catanese, M. Franciosi, K. Hulek and M. Reid.

**Theorem 1.4** ([7], Thm. 1.1). *Let  $C$  be a Cohen-Macaulay curve over an algebraically closed field  $k$ , let  $H$  be an invertible sheaf on  $C$ . Then*

- (1)  *$H$  is globally generated if for every generically Gorenstein subcurve  $B \subseteq C$ , we have  $H \cdot B \geq 2p_a(B)$ ;*
- (2)  *$H$  is very ample if for every generically Gorenstein subcurve  $B \subseteq C$ , we have  $H \cdot B \geq 2p_a(B) + 1$ .*

Recall that the arithmetic genus is defined by  $p_a(B) := 1 - \chi_k(\mathcal{O}_B)$  and that  $B$  is called generically Gorenstein if its dualizing sheaf is generically invertible. We will apply the above theorem to the closed fibres of  $f'$ . The key point is Corollary 2.10 which allows to express the arithmetic genus of an effective Weil divisor  $B$  on  $X'$  as the arithmetic genus of some effective Cartier divisor on  $X$ , to overcome the absence of the adjunction formula.

When  $S$  is of equal-characteristic (*i.e.*,  $\mathcal{O}_S$  contains a field), we provide in Section 4 a simpler proof of the main theorem based on I. Reider's method.

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## 2. ARITHMETIC GENUS OF WEIL DIVISORS

In this section we show how to compute the arithmetic genus of an effective Weil divisor on a normal surface with rational singularities. The main result is Corollary 2.10.

Let  $Y$  be a noetherian normal scheme. We first associate to each Weil divisor  $D \in Z^1(Y)$  a coherent sheaf  $\mathcal{O}_Y(D) \subseteq \mathcal{K}$ , where  $\mathcal{K}$  is the constant sheaf  $K(Y)$  of rational functions on  $Y$ .

**Definition 2.1** Let  $D \in Z^1(Y)$ . For any point  $\xi \in Y$  of codimension 1, denote by  $v_\xi$  the normalised discrete valuation on  $K(Y)$  induced by  $\xi$  and by  $v_\xi(D) \in \mathbb{Z}$  the multiplicity of  $D$  at  $\overline{\{\xi\}}$ . We define the sheaf  $\mathcal{O}_Y(D)$  by

$$\mathcal{O}_Y(D)(U) = \{x \in K(Y) \mid v_\xi(x) + v_\xi(D) \geq 0, \quad \forall \xi \in U, \text{ codim}(\xi, Y) = 1\}.$$

(By convention,  $v_\xi(0) = +\infty$ ).

**Remark 2.2** (1) Any  $\mathcal{O}_Y(D)$  is a reflexive sheaf by [13], Prop.1.6 since it is a normal sheaf.

(2) For any effective Weil divisor  $0 < D \in Z^1(Y)$ ,  $\mathcal{O}_Y(-D)$ , also denoted by  $\mathcal{I}_D$ , is an ideal sheaf. In this case, we consider  $D$  as the subscheme of  $Y$  defined  $\mathcal{I}_D$ .

(3) A purely codimension one closed subscheme  $Z$  of  $Y$  is an effective Weil divisor if and only if it has no embedded points.

**Lemma 2.3.** *Suppose  $D$  is an effective Weil divisor on  $Y$ . Let  $Y^0$  be an open subset of  $Y$  with complement of codimension at least 2, and let  $Z$  be any closed subscheme of  $Y$  such that  $D \cap Y^0 = Z \cap Y^0$ . Then  $\mathcal{I}_Z \subseteq \mathcal{I}_D$ .*

*Proof.* Let  $j : Y^0 \rightarrow Y$  be the canonical morphism. We have  $j^*\mathcal{I}_Z = j^*\mathcal{I}_D$  by hypothesis. Now the canonical injective morphism  $\mathcal{I}_Z \hookrightarrow j_*j^*\mathcal{I}_Z = j_*j^*\mathcal{I}_D$  implies  $\mathcal{I}_Z \subseteq \mathcal{I}_D$  because  $j_*j^*\mathcal{I}_D = \mathcal{I}_D$  ([13], Prop.1.6).  $\square$

From now on we focus on the case of surfaces. More precisely, we suppose  $Y = \text{Spec}(R)$  is an affine noetherian normal surface (*i.e.*, of dimension 2), with a unique singularity  $y \in Y$ . Suppose that  $Y$  admits a desingularization  $g : T \rightarrow Y$  (*i.e.*,  $T$  is regular,  $g$  is proper birational and  $g : T \setminus g^{-1}(y) \rightarrow Y \setminus \{y\}$  is an isomorphism). Let  $\{E_i\}_{i=1,\dots,m}$  be the set of prime divisors contained in  $g^{-1}(y)$ . We call an effective divisor  $Z$  on  $X$  exceptional if  $g(Z) = \{y\}$  as sets, *i.e.*,  $Z = \sum r_i E_i \geq 0$ .

**Definition 2.4** For any exceptional divisor  $Z$  and any coherent sheaf  $\mathcal{F}$  on  $Z$ , we define the characteristic of  $\mathcal{F}$ :

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \text{length}_R H^i(Z, \mathcal{F}).$$

We call  $p_a(Z) := 1 - \chi(\mathcal{O}_Z)$  the arithmetic genus of  $Z$ .

This definition makes sense since  $H^i(Z, \mathcal{F})$  is a finitely generated  $R$ -module supported on  $y$ , hence of finite length, and  $H^i(Z, \mathcal{F}) = 0$  as soon as  $i \geq 2$  since  $\dim Z \leq 1$ .

**Remark 2.5** In case  $T$  is an open subscheme of an arithmetic surface, and  $Z$  is supported in a special fibre, this definition coincides with [16], page 431, Equality (4.12).

We recall the definition of rational singularities for surfaces due to M. Artin :

**Proposition 2.6** ([1], see also [16], §9.4). *The following conditions are equivalent:*

- (1) *for any exceptional divisor  $Z$ , we have  $H^1(Z, \mathcal{O}_Z) = 0$ ;*
- (2) *for any  $Z$  as above,  $p_a(Z) \leq 0$ ;*
- (3)  *$R^1 g_*(\mathcal{O}_T) = 0$ .*

*When any of the above condition holds,  $y$  is called a rational singularity.*

**Remark 2.7** It is well known that the canonical model  $X'$  in Section 1 has only rational singularities ([16], Corollary 9.4.7).

From now on we assume  $y$  is a rational singularity. We need an important lemma.

**Lemma 2.8** ([15], Theorem 12.1). *For any  $\mathcal{L} \in \text{Pic}(T)$  such that  $\deg(\mathcal{L}|_{E_i}) \geq 0$  for all  $i \leq m$ ,  $\mathcal{L}$  is globally generated and  $H^1(T, \mathcal{L}) = 0$ .*  $\square$

**Proposition 2.9.** *Let  $B$  be an effective Weil divisor on  $Y$ , and let  $\tilde{B}$  be its strict transform in  $T$ . Then there is an exceptional divisor  $D$  on  $T$  such that  $\mathcal{I}_B \mathcal{O}_T = \mathcal{O}_T(-\tilde{B} - D)$ ,  $g_* \mathcal{O}_T(-\tilde{B} - D) = \mathcal{I}_B$  and  $R^1 g_* \mathcal{O}_T(-\tilde{B} - D) = 0$ .*

*Proof.* Let  $\mathcal{I} := \mathcal{I}_B \mathcal{O}_T$ . Let  $\Lambda$  be the set of exceptional divisors  $Z$  such that

$$(\tilde{B} + Z) \cdot E_i \leq 0, \quad \forall i \leq m.$$

Exactly as in [3], Lemma 3.18, one can prove that there is a smallest element  $D$  in  $\Lambda$ .

Let  $x \in \mathcal{I}_B \setminus \{0\}$ . Then  $\text{div}_T(x) = \tilde{B} + \tilde{C} + Z$ , where  $Z$  is exceptional and  $\tilde{C}$  is an effective divisor which does not contain any exceptional divisor. As  $\text{div}_T(x) \cdot E_i = 0$ , we have  $(\tilde{B} + Z) \cdot E_i = -E_i \cdot \tilde{C} \leq 0$ . So by definition  $Z \in \Lambda$  and  $\tilde{B} + Z \geq \tilde{B} + D$ ,

hence  $\mathcal{I} \subseteq \mathcal{O}_T(-\tilde{B} - D)$ . Since  $g_*(\mathcal{O}_T(-\tilde{B} - D))|_{Y \setminus \{y\}} = \mathcal{I}_B|_{Y \setminus \{y\}}$ , it follows from Lemma 2.3 that  $g_*\mathcal{O}_T(-\tilde{B} - D) \subseteq \mathcal{I}_B$ . Hence

$$\mathcal{I}_B \subseteq g_*\mathcal{I} \subseteq g_*\mathcal{O}_T(-\tilde{B} - D) \subseteq \mathcal{I}_B.$$

Therefore  $g_*\mathcal{O}_T(-\tilde{B} - D) = \mathcal{I}_B$ . By Lemma 2.8,  $g^*g_*\mathcal{O}_T(-\tilde{B} - D) \rightarrow \mathcal{O}_T(-\tilde{B} - D)$ , we have  $\mathcal{I} = \mathcal{O}_T(-\tilde{B} - D)$ . The same lemma implies that  $R^1g_*\mathcal{O}_T(-\tilde{B} - D) = H^1(T, \mathcal{O}_T(-\tilde{B} - D)) = 0$ .  $\square$

In the next corollary, we use the term “exceptional divisor on  $X$ ” in the sense of exceptional divisors for  $X \rightarrow X'$ .

**Corollary 2.10.** (1) *Let  $X, X', S$  be as in Section 1. Let  $B$  be an effective vertical Weil divisor on  $X'$ . Then there is an exceptional divisor  $D_B$  on  $X$  such that  $p_a(\tilde{B} + D_B) = p_a(B)$ .*  
 (2) *If  $M'$  is a normal proper algebraic surface over a field and with at worst rational singularities, and if  $g : M \rightarrow M'$  is any resolution of singularities, then for any effective Weil divisor  $B$  on  $M'$ , there is an exceptional divisor  $D_B$  such that  $p_a(\tilde{B} + D_B) = p_a(B)$ .*

*Proof.* (1) Let  $g : X \rightarrow X'$  be the canonical map. Let  $\mathcal{I}_B$  be the ideal sheaf on  $X'$  defined as in Remark 2.2 (3) and let  $\tilde{B}$  be the strict transform of  $B$  in  $X$ . By Proposition 2.9, there is an exceptional divisor  $D_B$  on  $X$  such that  $g_*\mathcal{O}_X(-\tilde{B} - D_B) = \mathcal{I}_B$  and  $R^1g_*\mathcal{O}_X(-\tilde{B} - D_B) = 0$ . Consider the exact sequence:

$$0 \rightarrow \mathcal{O}_X(-\tilde{B} - D_B) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\tilde{B}+D_B} \rightarrow 0.$$

Push-forwarding by  $g_*$ , we get:

$$0 \rightarrow \mathcal{I}_B \rightarrow \mathcal{O}_{X'} \rightarrow g_*\mathcal{O}_{\tilde{B}+D_B} \rightarrow R^1g_*\mathcal{O}_X(-\tilde{B} - D_B) = 0,$$

and

$$0 = R^1g_*\mathcal{O}_X \rightarrow R^1g_*\mathcal{O}_{\tilde{B}+D_B} \rightarrow R^2g_*\mathcal{O}_X(-\tilde{B} - D_B) = 0.$$

So  $g_*\mathcal{O}_{\tilde{B}+D_B} = \mathcal{O}_B$  and  $R^1g_*\mathcal{O}_{\tilde{B}+D_B} = 0$ . Hence  $\chi(\mathcal{O}_{\tilde{B}+D_B}) = \chi(\mathcal{O}_B)$ , and  $p_a(B) = p_a(\tilde{B} + D_B)$ .

(2) The proof is the same as for (1).  $\square$

### 3. PROOF OF THE MAIN THEOREM

Now we can prove the main theorem. We keep the notation of section 1.

**Lemma 3.1.** *For any vertical subcurve  $B_1$  of  $X'$ , there is a vertical effective Weil divisor  $B$  on  $X'$  and an open subset  $U' \subseteq X'$  with complement of codimension at least 2, such that  $B \cap U' = B_1 \cap U'$  and  $p_a(B) \geq p_a(B_1)$ .*

*Proof.* The curve  $B_1$  has only finitely many embedded points. Let  $U'$  be the complement of these points. Now as  $B_1 \cap U'$  do not admit any embedded points, it is an effective Weil divisor, and extends to a unique effective Weil divisor  $B$  on  $X'$ . By Lemma 2.3,  $\mathcal{I}_{B_1} \subseteq \mathcal{I}_B$  and, since the cokernel is clearly a skyscraper sheaf, we have  $\chi(\mathcal{O}_B) - \chi(\mathcal{O}_{B_1}) = -\chi(\mathcal{I}_B/\mathcal{I}_{B_1}) \leq 0$ , hence  $p_a(B) \geq p_a(B_1)$ .  $\square$

Since in our main theorem we do not assume that the generic fibre is geometrically connected, we need some more preparation. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ & \searrow g & \uparrow \pi \\ & & S' \end{array}$$

be the Stein factorization of  $f$ . For any  $s \in S$ , each connected component of  $X_s$  is equal set-theoretically to some fibers of  $X \rightarrow S'$ . The following proposition is well known:

**Proposition 3.2.** (1) *The dualizing sheaf  $\omega_{S'/S}$  is invertible, and we have  $\omega_{X/S} = g^* \omega_{S'/S} \otimes \omega_{X/S'}$ ;*  
 (2) *The arithmetic genus of (the generic fiber of)  $f$  is larger than 1 if and only if that of  $g$  is larger than 1.*  $\square$

**Lemma 3.3** ([14], Lemma 4). *Let  $s \in S$  be a closed point. Let  $\mathcal{L} \in \text{Pic}(X)$ . Then  $H^1(X_s, \mathcal{L}|_{X_s}) = 0$  if for any effective divisor  $0 < A \leq X_s$ , we have  $\mathcal{L} \cdot A > (\omega_{X/S} + A) \cdot A$ . In particular,  $H^1(X_s, \omega_{X_s/k(s)}^{\otimes n}) = 0$  whenever  $n \geq 2$ .*

*Proof.* The first part of the lemma is proved in [14]. We only have to show that  $\mathcal{L} := \omega_{X/S}^{\otimes n}$  satisfies the required condition for any  $n \geq 2$ . If  $\omega_{X/S}^{\otimes n} \cdot A \leq (\omega_{X/S} + A) \cdot A$ , then  $\omega_{X/S} \cdot A - A^2 \leq 0$  because  $\omega_{X/S} \cdot A \geq 0$  by the minimality of  $X \rightarrow S$ , hence  $\omega_{X/S} \cdot A = 0$ ,  $A^2 = 0$ , in particular  $A$  is the sum of some multiples of fibres of  $g$  (see the proof of [16], Theorem 9.1.23). But  $\omega_{X/S} \cdot X_{s'} > 0$  for any closed point  $s' \in S'$  by Proposition 3.2. Contradiction.  $\square$

*Proof of the main theorem.* First we observe that the statement is local on  $S$ , so we assume  $S$  is local. As the constructions of  $X$  and of  $X'$  commute with étale base changes, replacing  $S$  by its strict henselisation if necessary, we can assume that the residue field  $k$  at the closed point  $s \in S$  is algebraically closed. Here we use the hypothesis that  $k(s)$  is perfect.

By Lemma 3.3 and the upper-semicontinuity theorem ([12], III.12), for any  $n \geq 2$ , we have  $R^1 f_*(\omega_{X/S}^{\otimes n}) = 0$ . So  $R^1 f'_*(\omega_{X'/S}^{\otimes n}) = 0$  since  $\pi_* \omega_{X/S}^{\otimes n} = \omega_{X'/S}^{\otimes n}$  and

$$R^1 \pi_* \omega_{X/S}^{\otimes n} = R^1 \pi_*(\pi^* \omega_{X'/S}^{\otimes n}) = R^1 \pi_* \mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \omega_{X'/S}^{\otimes n} = 0.$$

Again by the upper-semicontinuity theorem we have  $H^1(X'_s, \omega_{X'/S}^{\otimes n}) = 0$ , and the canonical morphism  $H^0(X', \omega_{X'/S}^{\otimes n}) \otimes k \rightarrow H^0(X'_s, \omega_{X'_s/k}^{\otimes n})$  is an isomorphism.

Let  $n \geq 3$ . Then it is enough to prove that  $\omega_{X'/S}^{\otimes n}|_{X'_s} = \omega_{X'_s/k}^{\otimes n}$  is very ample. Let  $C = X'_s$ . This is a Cohen-Macaulay curve over  $k$ . We will show for any subcurve  $B_1$  of  $C$  that

$$n(\omega_{X'_s/k} \cdot B_1) \geq 3(\omega_{X'_s/k} \cdot B_1) \geq 2p_a(B_1) + 1.$$

Theorem 1.4 (2) will then imply that  $\omega_{X'_s/k}^{\otimes n}$  is very ample. Let  $B$  be the effective Weil divisor on  $X'$  as given by Lemma 3.1. By construction,  $p_a(B) \geq p_a(B_1)$ . As  $B$  and  $B_1$  differ at worst by a zero-dimensional closed subset, we have  $\omega_{X'_s/k} \cdot B = \omega_{X'_s/k} \cdot B_1$  (use e.g. [10], Prop. 6.2). Therefore it is enough to show the desired inequality for  $B$ .

By Corollary 2.10,  $p_a(B) = p_a(\tilde{B} + D)$  for some exceptional divisor  $D$ . Note in our case that  $\omega_{X'_s/k} \cdot B = \omega_{X'/S} \cdot B = \omega_{X/S} \cdot \tilde{B} = \omega_{X/S} \cdot (\tilde{B} + D)$  since  $\omega_{X/S} \cdot D = 0$  by our assumption. So we need to prove:

$$3\omega_{X/S} \cdot (\tilde{B} + D) \geq 2p_a(\tilde{B} + D) + 1 = 3 + (\tilde{B} + D)^2 + \omega_{X/S} \cdot (\tilde{B} + D),$$

i.e.,

$$2\omega_{X/S} \cdot (\tilde{B} + D) - (\tilde{B} + D)^2 \geq 3.$$

This is true since  $\omega_{X/S} \cdot (\tilde{B} + D) \geq 1$ ,  $-(\tilde{B} + D)^2 \geq 0$ , and if the left-hand side is equal to 2, then  $\omega_{X/S} \cdot (\tilde{B} + D) = 1$  and  $(\tilde{B} + D)^2 = 0$ , which is impossible as  $(\tilde{B} + D)^2 + \omega_{X/S} \cdot (\tilde{B} + D)$  is always even.  $\square$

**Proposition 3.4** ([14], Theorem 7). *Keep the notation of Theorem 1.2 and suppose  $S$  is affine. Then  $\omega_{X/S}^{\otimes n}$  is globally generated for all  $n \geq 2$ .*

*Proof.* Keep the notation of the above proof. Then it is enough to prove that  $\omega_{X_s/k}^{\otimes n}$  is globally generated. By Theorem 1.4 (1), it is enough to prove that  $\omega_{X/S}^{\otimes n} \cdot B \geq 2p_a(B)$  for all  $0 < B \leq X_s$ , or that

$$2\omega_{X/S} \cdot B \geq 2 + (\omega_{X/S} \cdot B + B^2)$$

which is equivalent to

$$\omega_{X/S} \cdot B - B^2 \geq 2.$$

It is clear that the left-hand side is a non-negative even number, and it is zero only if both  $\omega_{X/S} \cdot B = 0$  and  $B^2 = 0$ . But this is impossible as we pointed out in the proof of Lemma 3.3.  $\square$

#### 4. PROOF OF THE MAIN THEOREM IN THE EQUAL-CHARACTERISTIC CASE

In this section we give an alternative proof of the main theorem under the assumption that  $S$  is of equal-characteristic (i.e.,  $\mathcal{O}_S$  contains a field). We keep the notation of Section 1. Using Lemma 3.3 and the upper semi-continuity theorem ([12], III.12), we find that it is sufficient to prove  $\omega_{X'_s/k(s)}^{\otimes n}$  is very ample for  $n \geq 3$ . Note that this conclusion in fact depends only on  $X_s$ : the pluri-canonical morphism restricted to the special fibre is exactly that defined by  $H^0(X_s, \omega_{X_s/k}^{\otimes n})$ , and  $X'_s = \text{Proj}(\bigoplus_i H^0(X_s, \omega_{X_s/k}^{\otimes i}))$  is canonically determined by  $X_s$ . In particular, to prove the main theorem we are able to interchange  $X$  with another minimal arithmetic surface which has the same special fibre.

The next lemma will allow us to reduce to the case when  $S$  is a curve over a field.

**Lemma 4.1.** *Suppose  $S = \text{Spec} R$  with  $R = k[[t]]$  and  $k$  algebraically closed. Let  $s$  be the closed point of  $S$ . Then there exists a minimal fibration  $h : Y \rightarrow C$  with  $Y$  an integral projective smooth surface of general type over  $k$ ,  $C$  an integral projective smooth curve over  $k$  of genus  $g \geq 2$ , and a closed point  $c \in C$  such that  $X_s$  is isomorphic to  $Y_c$  as  $k$ -schemes.*

*Proof.* Let  $A$  be the Henselisation of  $k[t]_{t \nmid k[t]}$ . Then  $R$  is the completion of  $A$ . In particular  $k[t]/t^2 = R/t^2R = A/t^2A$ . Let  $S_1$  and  $V$  denote the spectra of  $k[t]/t^2$  and of  $A$  respectively. Thus  $\{s\} = \text{Spec} k$  and  $S_1$  are considered as closed subschemes of both  $S$  and  $V$ .

As  $f : X \rightarrow S$  is flat and projective, it descends to a flat projective scheme  $W \rightarrow T$  with  $T$  integral of finite type over  $k$ , [11], IV.11.2.6. By Artin approximation theorem in [1], there is a morphism  $\varphi : V \rightarrow T$ , such that the diagram below commutes:

$$\begin{array}{ccc} S_1 & \hookrightarrow & V \\ \downarrow & & \downarrow \varphi \\ S & \longrightarrow & T \end{array}$$

We claim that  $Z = W \times_T V$  is regular. Indeed,  $Z_s \simeq X_s$  as  $s$  is a closed subscheme of  $S_1$  and of  $X \times_S S_1 = W \times_T S_1 = Z \times_V S_1$ . With this identification, for any closed point  $p \in Z_s$ , we have  $T_{Z,p} = T_{Z \times_V S_1,p} = T_{X \times_S S_1,p} = T_{X,p}$ , where  $T_{?,?}$  denotes the Zariski tangent space. As  $\dim_k T_{X,p} = 2$  because  $X$  is regular, we have  $\dim_k T_{Z,p} \leq 2$ . By the flatness of  $Z/V$ ,  $Z$  is a 2-dimensional scheme. So  $Z$  is regular. Note also that  $Z \rightarrow V$  is relatively minimal as this property can be checked in the closed fiber and  $Z_s \simeq X_s$ .

By the construction of  $V$ ,  $Z \rightarrow V$  descends to a relatively minimal arithmetic surface  $Y_1 \rightarrow C_1$  where  $C_1$  is an integral affine smooth curve over  $k$ . After compactifying  $C_1$  and  $Y_1$ , we find a minimal fibration  $Y \rightarrow C$  as desired. We take the point  $c \in C$  to be the image of  $s \in V$  in  $C_1 \subset C$ . Finally replacing  $C$  by some finite cover that is étale at  $c$  if necessary, we may assume  $g(C) \geq 2$ , and consequently  $Y$  is of general type ([8] Theorem 1.3).  $\square$

Note that when  $g(C) \geq 1$ , the relative canonical model  $X/C$  coincides with the canonical model of  $X$ .

**Theorem 4.2.** *Let  $S, X, X'$  be as in Section 1. Suppose further that  $S$  is an integral projective smooth curve of genus  $\geq 2$  over an algebraically closed field  $k$ . Let  $K_{X'}$  be a canonical divisor of  $X'$  over  $k$ . Then for any sufficiently ample divisor  $M$  on  $S$ ,  $nK_{X'} + f'^*M$  is very ample on  $X'$  if  $n \geq 3$ .*

*Proof.* By Reider's method (see Corollary 5.2(3)), we just need to prove that  $H^1(X, nK_X + f^*M - 2Z) = 0$ , where  $Z$  is the vertical fundamental cycle of the  $(-2)$ -curves lying above a singularity of  $X'$ .

By our assumption on  $M$  (being sufficient ample) and applying spectral sequence, it is sufficient to show  $R^1 f_* \mathcal{O}_X(nK_X + f^*M - 2Z) = 0$  or, equivalently, that  $H^1(X_s, \mathcal{O}_X(nK_X - 2Z)|_{X_s}) = 0$  for any closed point  $s \in S$  (note that  $\mathcal{O}_X(f^*M)|_{X_s}$  is trivial). As  $\mathcal{O}_X(K_X)|_{X_s} \simeq \omega_{X/S}|_{X_s}$ , by Lemma 3.3 we only need to consider the case when  $s$  is the image of  $Z$  in  $S$ . Again by Lemma 3.3, to prove the vanishing above  $s$ , it is enough to show that for any divisor  $0 < A \leq X_s$ ,

$$((n-1)K_X - 2Z) \cdot A > A^2.$$

Suppose the contrary. Then

$$2 \geq (A + Z)^2 + 2 \geq (n-1)K_X \cdot A \geq 2K_X \cdot A \geq 0$$

(note that  $Z^2 = -2$ ). This is impossible:

- (i) if  $K_X \cdot A = 0$ , then  $A$  consists of  $(-2)$ -curves, in particular  $Z \cdot A \leq 0$  by the definition of the fundamental cycle, so  $(A + Z)^2 \leq A^2 + Z^2 < -2$ , contradiction;
- (ii) if  $K_X \cdot A = 1$ , then  $(A + Z)^2 = 0$ , so  $(A + Z) \cdot B = 0$  for any vertical divisor  $B$  and thus  $A^2 = ((A + Z) - Z)^2 = -2$ . This implies that  $K_X \cdot A + A^2 = -1$  is odd, contradiction.  $\square$



Now we can proceed to the proof of our main results.

*Proof of Theorem 1.2 and of Proposition 3.4.* Similarly to the previous section, we can assume  $S = \text{Spec} R$  with  $R = k[[t]]$  and  $k$  algebraically closed. Using Lemma 4.1, we can descend  $X \rightarrow S$  and suppose that  $S$  is an integral projective smooth curve over  $k$  of genus  $g \geq 2$ . Let  $M$  be a sufficiently ample divisor on  $S$ . By Theorem 4.2,  $nK_{X'} + f'^*M$  is very ample for any  $n \geq 3$ . Therefore  $\omega_{X'_s/k}^{\otimes n} \simeq \mathcal{O}_{X'}(nK_{X'} + f'^*M)|_{X'_s}$  is also very ample. Similarly, if  $n \geq 2$ ,  $\omega_{X'_s/k}^{\otimes n}$  is globally generated using Corollary 5.2 (1).  $\square$

**Remark 4.3** One can also prove Theorem 1.2 under the assumptions of 4.2 using Theorem 1.2 of [7]. Indeed, replacing  $C$  by a finite étale cover of sufficiently high degree if necessary, we may assume  $\chi(\mathcal{O}_X) \neq 1$ , and  $g(C) \gg 0$  so that  $p_g(X) \geq 2$  (if  $\chi(\mathcal{O}_X) \geq 0$ , then  $p_g(X) \geq g(X) - 1 \geq g(C) - 1 \gg 2$ ; if  $\chi(\mathcal{O}_X) < 0$ , then  $p_g(X) \geq -2\chi(\mathcal{O}_X) \geq 2$  by [18], Lemma 13). One then checks that the conditions of [7], Theorem 1.2 are satisfied. Therefore  $|nK_{X'}|$  is very ample if  $n \geq 3$ . In particular  $\omega_{X'_s/k}^{\otimes n} \simeq \mathcal{O}_{X'}(nK_{X'})|_{X'_s}$  is also very ample.

## 5. REIDER'S METHOD

Below is the statement of Reider's method ([17]; [4], p. 176) in any characteristic.

**Theorem 5.1** (Reider's Method). *Let  $X$  be an integral projective smooth surface over an algebraically closed field, and let  $L$  be a nef divisor on  $X$ .*

- (1) *Suppose that any vector bundle  $E$  of rank 2 with  $\delta(E) := c_1^2(E) - 4c_2(E) \geq L^2 - 4$  is unstable in the sense of Bogomolov ([5]; [4], p. 168). If  $P$  is a base point of  $|K_X + L|$ , then there is an effective divisor  $D$  passing through  $P$  such that*
  - (a)  $D \cdot L = 0$  and  $D^2 = -1$ ; or
  - (b)  $D \cdot L = 1$  and  $D^2 = 0$ .
- (2) *Suppose that any vector bundle  $E$  of rank 2 with  $\delta(E) \geq L^2 - 8$  is unstable in the sense of Bogomolov. If  $|K_X + L|$  fails to separate  $P, Q$  (possibly infinite near), then there is an effective divisor  $D$  passing through  $P, Q$  such that*
  - (a)  $D \cdot L = 0$  and  $D^2 = -2$  or  $-1$ ; or
  - (b)  $D \cdot L = 1$  and  $D^2 = 0$  or  $-1$ ; or
  - (c)  $D \cdot L = 2$  and  $D^2 = 0$ ; or
  - (d)  $L^2 = 9$  and  $L$  numerically equivalent to  $3D$ .

$\square$

**Corollary 5.2.** *Keep the above notation and suppose there exists a relatively minimal fibration  $f : X \rightarrow S$  whose fibres have arithmetic genus  $\geq 2$  and such that  $g(S) \geq 2$ . Then:*

- (1)  $|nK_X + f^*M|$  is base point free if  $n \geq 2$  and  $M$  is sufficiently ample.
- (2) When  $n \geq 3$  and  $M$  is sufficiently ample,  $|nK_X + f^*M|$  is very ample outside the locus of vertical  $(-2)$ -curves and also separates points not connected by vertical  $(-2)$ -curves.
- (3) Keep the hypothesis of (2). Let  $f' : X' \rightarrow S$  be the canonical model of  $X$  and let  $K_{X'}$  be a canonical divisor on  $X'$ . Suppose that for the fundamental cycle  $Z \subset X$  above any singular point of  $X'$  we have

$$H^1(X, \mathcal{O}_X(nK_X + f^*M - 2Z)) = 0,$$

then  $nK_{X'} + f'^*M$  is very ample.



*Proof.* (1) - (2) Let  $L = (n - 1)K_X + f^*M$ , so

$$L^2 = (n - 1)^2 K_X^2 + 2(n - 1)(2g - 2) \deg M \gg 0, \quad \text{if } \deg M \gg 0.$$

In characteristic 0, it is well known that any  $E$  with  $\delta(E) > 0$  is unstable. In positive characteristic  $p$  case, we apply [18], Theorem 15, which says that  $E$  is semi-stable only if either  $pK_X^2 \geq p/2(\sqrt{K_X^2 \cdot \delta}) + 2\chi(\mathcal{O}_X) + p(2p - 1)\delta/6$  or  $K_X^2 \geq \delta$ . So anyway, when  $M$  is sufficiently ample, the instability conditions on vector bundles in the above theorem hold. Then by standard discussions we can prove the conditions (a)(b) in 5.1(1) will not occur and in 5.1(2) only condition (a) where  $L \cdot D = 0, D^2 = -2$  can occur, in this case  $D$  is a sum of  $(-2)$ -curves. So (1), (2) are proved.

(3) We have  $H^0(X', \mathcal{O}_{X'}(nK_{X'} + f'^*M)) = H^0(X, \mathcal{O}_X(nK_X + f^*M))$  and the map  $X \rightarrow \phi_{|nK_X + f^*M|}$  factors through  $h : X' \rightarrow \phi_{|nK_X + f^*M|}$ . Under the assumption of (2),  $h$  is a homeomorphism and is an isomorphism outside the singularities of  $X'$ . In order to prove it is an isomorphism it is sufficient to prove that  $|nK_{X'} + f'^*M|$  separates tangent spaces of each singularities of  $X'$  (see also [6] p. 72). Let  $x$  be such a singularity and  $Z$  be the fundamental cycle of  $(-2)$ -curves lying above  $x$ . So what we need is to prove the following sequence is exact:

$$\begin{aligned} 0 \rightarrow H^0(X', \mathfrak{m}_x^2 \mathcal{O}_{X'}(nK_{X'} + f'^*M)) &\rightarrow H^0(X', \mathfrak{m}_x \mathcal{O}_{X'}(nK_{X'} + f'^*M)) \\ &\rightarrow H^0(X', \mathfrak{m}_x / \mathfrak{m}_x^2 \otimes \mathcal{O}_{X'}(nK_{X'} + f'^*M)) \rightarrow 0 \end{aligned}$$

where  $\mathfrak{m}_x$  denotes the maximal ideal of  $\mathcal{O}_{X',x}$ . It is enough to show that

$$H^1(X', \mathfrak{m}_x^2 \mathcal{O}_{X'}(nK_{X'} + f'^*M)) = 0.$$

Let  $\pi$  denote the canonical morphism from  $X$  to  $X'$ , then it is well known that  $\pi_* \mathcal{O}_X(nK_X + f^*M - 2Z) = \mathfrak{m}_x^2 \mathcal{O}_{X'}(nK_{X'} + f'^*M)$  ([3], Thm. 3.28). For any irreducible component  $E$  of the exceptional locus of  $X \rightarrow X'$ , we have

$$E \cdot (nK_X + f^*M - 2Z) = nE \cdot K_X - 2E \cdot Z = nE \cdot \omega_{X/S} - 2E \cdot Z \geq 0$$

by the minimality of  $X \rightarrow S$  and because  $Z$  is a fundamental cycle. Therefore it follows from Lemma 2.8 that  $R^1 \pi_* \mathcal{O}_X(nK_X + f^*M - 2Z) = 0$ , hence

$$H^1(X', \mathfrak{m}_x^2 \mathcal{O}_{X'}(nK_{X'} + f'^*M)) = H^1(X, \mathcal{O}_X(nK_X + f^*M - 2Z)) = 0$$

by assumption.  $\square$

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